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Engineering Notes

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Parameter Optimal Control of Wing-Rock

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I. Introduction

RECENTLY, an optimal control of wing-rock was described.[1] In Ref. 1, the optimal control time function obtained was least-squares fitted into a specified form of control. This Note presents an algorithm[2] for determining the optimal constant coefficients directly.

II. Background

The system to be controlled is in general described by the nonlinear state vector equation

$$\dot{x} = f(x, u, t) \tag{1}$$

The optimal control is the function $u(t)$ that minimizes the performance index

$$J = \theta[x(t_f), t_f] + \int_{t_0}^{t_f} \phi[x(t), u(t), t]\, dt \tag{2}$$

subject to the constraint stated by Eq. (1).

Following Bryson and Ho,[3] the solution of this optimal control problem is based on the definition of the scalar Hamiltonian

$$H = \phi[x(t), u(t), t] + \lambda^T(t) f[x(t), u(t), t] \tag{3}$$

The necessary condition for a minimum of the performance index, given by Eq. (2), is that its first variation be zero. Setting this variation to zero yields

$$\left[\left(\frac{\partial \theta}{\partial x} - \lambda^T\right)\delta x\right] = 0, \qquad t = t_f \tag{4}$$

$$[\lambda^T \delta x] = 0, \qquad t = t_0 \tag{5}$$

$$-\dot{\lambda}^T = \frac{\partial H}{\partial x} = \frac{\partial \phi(x, u, t)}{\partial x} + \lambda^T(t)\frac{\partial f(x, u, t)}{\partial x} \tag{6}$$

For the conventional optimal control problem, it is usually assumed that the initial conditions for Eq. (1) are specified. The terminal conditions for this problem are then obtained from Eq. (4) as

$$\lambda^T(t_f) = \frac{\partial \theta[x(t_f), t_f]}{\partial x(t_f)} \tag{7}$$

If, however, some of the initial and final state vector elements are not specified, then Eqs. (4) and (5) are used to determine the terminal conditions.

III. Parameter Optimal Control

For parameter optimal control, it is desired to form the control vector $u(t)$ as a function of the states x, control parameters a, and time t as

$$u = g(x, a, t) \tag{8}$$

The state equation (1) is then reformulated and rewritten as

$$\dot{x} = f[x, g(x, a, t), t] \tag{9}$$

and the parameter vector satisfies

$$\dot{a} = \mathbf{0} \tag{10}$$

The performance index in Eq. (1) is then modified to reflect the specific form of the control given in Eq. (8). The objective of the parameter optimal control problem is to determine the values of the parameter vector a that minimize the modified performance index subject to the constraints given in Eqs. (9) and (10).

The solution approach for the parameter optimal control problem presented in this Note is based on a reformulation of the states and makes use of the equations developed above for the conventional optimal control problem. This reformulation adjoins the parameter state [Eq. (10)] to the system state [Eq. (9)] to form a new state vector. This reformulated system state vector and the corresponding Lagrange multiplier can be represented as

$$\chi = \begin{bmatrix} x \\ a \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \lambda \\ \Gamma \end{bmatrix} \tag{11}$$

Equation (6) is now used to describe the reformulated state vector's Lagrange multiplier with Λ replacing λ. The Lagrange multiplier initial and terminal conditions for the reformulated state-vector elements, corresponding to the parameter elements of χ, are

$$\Gamma(t_0) = \mathbf{0}, \qquad \Gamma(t_f) = \mathbf{0} \tag{12}$$

Equation (7) is used to specify terminal conditions for Lagrange multipliers corresponding to the original state-vector elements.

The solution for the parameter vector a is accomplished by iteration using the algorithm

$$a_{i+1} = a_i - K\left[\frac{\partial \Gamma(t_0)}{\partial a}\right]_i^{-1} \Gamma_i(t_0) \tag{13}$$

The application of this algorithm requires the influence function $\partial\Gamma/\partial a$. In this Note, this influence function is obtained from numerical differencing perturbation solutions about a trial solution.

The algorithm in Eq. (13) proceeds by the following steps:

1) With assumed values for the parameter states, integrate the state equation (9) forward to the final time $t = t_f$.

2) Use the resulting state values to initialize the state Lagrange multiplier equation (7).

3) Integrate, in reverse time, the Lagrange multiplier equations, state and parameter, to the initial time $t = t_0$.

4) Compute the inverse indicated in Eq. (13) and update the control parameter state-vector elements.

Iterations continue until values of the parameter's Lagrange multiplier $\Gamma_i(t_0)$ are sufficiently small.

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Table 1 Index weightings,[a] computed control constants, and costs

Case	b_1, s	b_2, s^{-1}	k_1, s^{-2}	k_2, s^{-1}	k_3, rad^{-1}-s^{-2}	k_4, rad^{-1}	k_5, rad^{-1}-s^{-1}	k_6, rad^{-1}-s^{-1}	J, rad^2
2	1.71	2.55	−0.00129	−2.3035	−1.4409	0.28091	1.1255	0.60540	0.584775
7	25.11	3.39	−0.01585	−5.8090	−0.77455	−0.13620	1.8867	0.23427	1.52847
2[b]	1.71	2.55	−0.05395	−2.1998	—	—	—	—	0.594890
7[b]	25.11	3.39	−0.24308	−5.7988	—	—	—	—	1.52849

[a]Index weightings are from Ref. 1. [b]Control using k_1 and k_2 only.

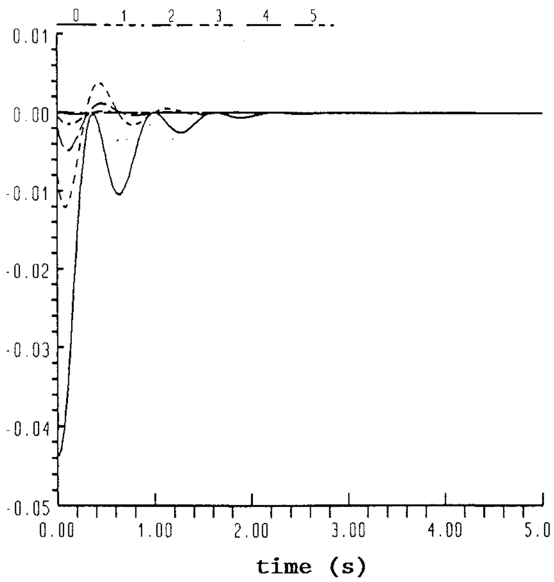

Fig. 1 Lagrange multiplier iterations.

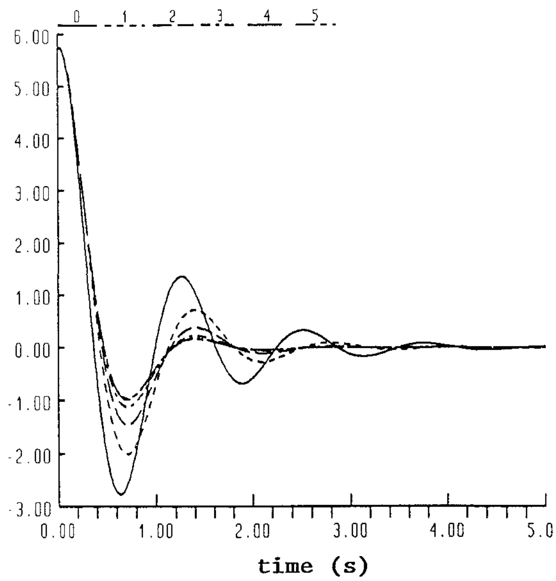

Fig. 2 Roll angle iterations (degrees).

IV. Wing-Rock Control Results

Wing-rock is described in Ref. 1 [Eq. (2)] by the following nonlinear differential equations:

$$\dot{\phi} = p \tag{14}$$

$$\dot{p} = c_0 + c_1\phi + c_2 p + c_3 p\phi \operatorname{sgn}(\phi) + c_4 p^2 \operatorname{sgn}(p) + u \tag{15}$$

where ϕ is the roll angle and p is the roll rate. The control u is determined by minimizing the performance index

$$J = \int_{t_0}^{t_f} (b_1 p^2 + b_2\phi^2 + b_3 u^2)\, dt \tag{16}$$

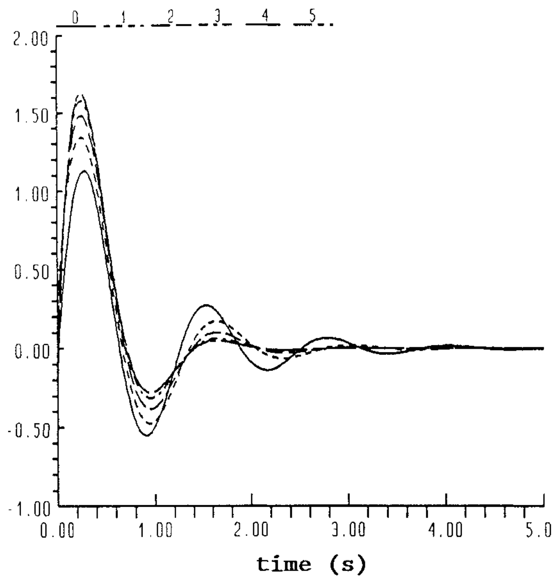

Fig. 3 Control iterations.

The specified form for the control in Ref. 1 [Eq. (56)] is

$$u = k_1\phi + k_2 p + k_3\phi^2 \operatorname{sgn}(\phi) + k_4 p^2 \operatorname{sgn}(p) + k_5 p\phi \operatorname{sgn}(\phi) + k_6 p\phi \operatorname{sgn}(p) \tag{17}$$

The numerical values for the nonzero c, used in describing wing-rock dynamics in Eq. (15), are summarized as

$$c_1 = -26.6667\ s^{-2}, \qquad c_2 = 0.76485\ s^{-1}, \qquad c_3 = -2.92173\ rad^{-1}\text{-}s^{-1} \tag{18}$$

and the initial conditions for the states in Eqs. (14) and (15) are

$$\phi(t_0) = 0.1\ \text{rad}, \qquad p(t_0) = 0.032\ \text{rad/s} \tag{19}$$

Two cases are examined from Ref. 1. These cases are defined by the performance index weightings in Eq. (16) and are presented in Table 1. The value of b_3 in Eq. (16) for all cases presented is 1.0. Also listed in Table 1 are the values of the k resulting from the algorithm in Eq. (13). In addition to these two cases from Ref. 1, two additional cases are examined with the form of the control given in Eq. (17) changed to include only the first two terms. The initial parameter values used to start the algorithm in Eq. (13) are all zero except k_2, which is initialized with the value of −3.0.

The numerical values in Table 1 for the two versions of case 7 suggest that little additional costs are incurred when the control uses only the first two terms in Eq. (17) and only a slight increase in costs is seen for the two versions of case 2. To illustrate the convergence of the algorithm in Eq. (13), the k_1 Lagrange multiplier state trajectories are presented in Fig. 1 for the iterations of the algorithm in Eq. (13). The roll angle ϕ and control u resulting from these iterations are presented in Figs. 2 and 3, respectively.

V. Conclusions

An algorithm has been presented to compute the optimal parameters for a specified form of control for nonlinear systems and has been applied to the optimal control of wing-rock.

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Yaw Pointing/Lateral Translation Using Robust Sampled Data Eigenstructure Assignment

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Introduction

SOBEL and Shapiro[1] used eigenstructure assignment to design a continuous-time controller for the yaw pointing/lateral translation maneuver of the Flight Propulsion Control Coupling (FPCC) aircraft. The design of Sobel and Shapiro[1] is characterized by perfect decoupling, but the minimum of the smallest singular value of the return difference matrix at the aircraft inputs was only 0.18.

Sobel and Lallman[2] proposed a pseudocontrol strategy for reducing the dimension of the control space by using singular-value decomposition. The FPCC yaw pointing/lateral translation design of Sobel and Lallman[2] yields a minimum of the smallest singular value of the return difference matrix at aircraft inputs of 0.9835, but the lateral translation transient response has significant coupling to the heading angle.

Sobel and Shapiro[3] have proposed an extended pseudocontrol strategy. Piou and Sobel[4] extended eigenstructure assignment to linear time-invariant plants that are represented by the unified delta model,[5] which is valid both for continuous-time and sampled-data operation of the plant. Piou et al.[6] have extended Yedavalli's[7] Lyapunov approach for stability robustness of a linear time-invariant system to the unified delta system.[5] In this Note, the results of Refs. 3, 4, and 6 are used to design a robust sampled-data extended pseudocontrol eigenstructure assignment flight control law for the yaw pointing/lateral translation maneuver of the FPCC aircraft.

Problem Formulation

Consider a nominal linear time-invariant system described by (A, B, C). The corresponding sampled-data system[4] is described by (A_δ, B_δ, C) and the unified delta model[4] is described by (A_ρ, B_ρ, C). Suppose that the nominal delta system is subject to linear time-invariant uncertainties in the entries of A_ρ and B_ρ, described by $\mathrm{d}A_\rho$ and $\mathrm{d}B_\rho$, respectively. Then, the delta system with uncertainty is given by $(A_\rho + \mathrm{d}A_\rho, B_\rho + \mathrm{d}B_\rho, C)$. Here $\mathrm{d}A_\rho = \mathrm{d}A, \mathrm{d}B_\rho = \mathrm{d}B$ in continuous time and $\mathrm{d}A_\rho = \mathrm{d}A_\delta, \mathrm{d}B_\rho = \mathrm{d}B_\delta$ in discrete time. Furthermore, suppose that bounds are available on the maximum absolute values of the elements of $\mathrm{d}A$ and $\mathrm{d}B$. Define $\mathrm{d}A^+$ and $\mathrm{d}B^+$ as the matrices obtained by replacing the entries of $\mathrm{d}A$ and $\mathrm{d}B$ by their absolute values, respectively. Also, define $A_{\max}$ and $B_{\max}$ as the matrices whose entries are the element-by-element bounds on the absolute values of the entries of $\mathrm{d}A$ and $\mathrm{d}B$, respectively. Then, $\{\mathrm{d}A : \mathrm{d}A^+ \le A_{\max}\}$ and $\{\mathrm{d}B : \mathrm{d}B^+ \le B_{\max}\}$ where $\le$ is applied element by element to matrices.

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Consider the constant gain output feedback control law described by $u(t) = F_\rho y(t)$ where $F_\rho = F$ in continuous time and $F_\rho = F_\delta$ in discrete time. Then, the nominal closed-loop unified delta system is given by $\rho x(t) = A_{\rho c} x(t)$ where $A_{\rho c} = A + BFC$ in continuous time and $A_\delta + B_\delta F_\delta C$ in discrete time. The uncertain closed-loop unified delta system is given by $\rho x(t) = A_{\rho c} x(t) + \mathrm{d}A_{\rho c} x(t)$ where $\mathrm{d}A_{\rho c} = \mathrm{d}A + \mathrm{d}B(FC)$ in continuous time and $\mathrm{d}A_\delta + \mathrm{d}B_\delta(F_\delta C)$ in discrete time. The reader is referred to Ref. 4 for a more detailed description.

Pseudocontrol and Robustness Results

Consider the singular-value decomposition of the matrix B_ρ given by

$$B_\rho = [U_1 \quad U_2 \quad U_0] \begin{bmatrix} \Sigma_1 & & \\ & \Sigma_2 & \\ & & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \\ V_0^T \end{bmatrix} \tag{1}$$

where $\Sigma_1 = \mathrm{diag}[\sigma_1, \ldots, \sigma_a]$ and $\Sigma_2 = \mathrm{diag}[\sigma_{a+1}, \ldots, \sigma_b]$ and $\sigma_b \le \sigma_{b-1} \le \cdots \le \sigma_{a+1} \le \epsilon$ with ϵ small.

Lemma.[3] Let the system with the pseudocontrol $\delta(t)$ be described by

$$\rho x(t) = A_\rho x(t) + \tilde{B}_\rho \delta(t) \tag{2}$$

$$y(t) = Cx(t) \tag{3}$$

where

$$\tilde{B}_\rho = U_1 + U_2[\alpha_1, \ \alpha_2] \tag{4}$$

We design a feedback pseudocontrol for the system described by Eqs. (2–4).

Then, the true control $u(t)$ for the system described by (A_ρ, B_ρ, C) is given by

$$u(t) = \left[V_1 \Sigma_1^{-1} + V_2 \Sigma_2^{-1} \alpha\right] \delta(t) \tag{5}$$

Furthermore, when $\alpha = [0, 0]^T$, the control law $u(t)$ given by Eq. (5) reduces to the control law given by Eq. (20) in Ref. 2.

Theorem.[6] The system matrix $A_{\rho c} + \mathrm{d}A_{\rho c}$ is stable if

$$\sigma_{\max}\left(E_{2\max}^T P_\rho^+ E_{1\max}\right)_s < 1 \tag{6}$$

where

$$E_{1\max} = A_{\rho\max} + B_{\rho\max}(F_\rho C)^+$$

$$E_{2\max} = \left\{I_n + \Delta\left[A_\rho + B_\rho(F_\rho C)\right]\right\}^+ + (\Delta/2)E_{1\max}$$

and where P_ρ satisfies the Lyapunov equation given by

$$A_{\rho c}^T P_\rho + P_\rho A_{\rho c} + \Delta A_{\rho c}^T P_\rho A_{\rho c} = -2I_n$$

P_ρ^+ is the matrix formed by the modulus of the entries of the matrix P_ρ, and $(\cdot)_s$ denotes the symmetric part of a matrix.

Yaw Pointing/Lateral Translation Control Law Design

We consider the FPCC aircraft linearized lateral dynamics described in Ref. 2. The objective of yaw pointing control is to command the aircraft yaw heading without a change in the lateral flight path angle or bank angle.

The objective of lateral translation control is to command the lateral flight path angle without a change in yaw heading or bank angle. First, we design an eigenstructure assignment control law by using an orthogonal projection. The delta state-space matrices A_δ and B_δ are computed by using the MATLAB Delta Toolbox.[8] The sampling period Δ is chosen to be 0.02 s for illustrative purposes. The achievable eigenvectors are computed by using the orthogonal projection of the ith desired eigenvector v_i^d onto the subspace spanned by the columns of $(\gamma_i I - A_\delta)^{-1} B_\delta$. The desired eigenvalues are $\gamma_{\mathrm{dr}} = (1/\Delta)\{\exp[(-2 \pm j2)\Delta] - 1\}$ for the dutch roll mode, $\gamma_{\mathrm{roll}} = (1/\Delta)\{\exp[(-3 \pm j2)\Delta] - 1\}$ for the roll mode, and